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THE APPROXIMATION PROPERTY AND THE CHAIN CONDITION

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1. The approximation property

Definition 1.1. Let $\mathbb{P}$ a be poset and κ a cardinal. We say that the poset $\mathbb{P}$ has the *κ-approximation property* if for every ordinal τ and every $f \in ({}^{\tau}2)^{V^{\mathbb{P}}}$, if $f|x \in V$ for every $x \in ([\tau]^{<\kappa})^V$, then $f \in V$.

It is known that for an uncountable κ, if $\mathbb{P}$ is an atomless poset of size $< \kappa$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a κ-closed poset, then $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-approximation property (e.g., see Mitchell [1]). In this note , we show that the size assumption for a poset $\mathbb{P}$ can be relaxed to the chain condition assumption.

Definition 1.2. Let κ be a regular uncountable cardinal. A poset $\mathbb{P}$ satisfies the *strong κ-chain condition* (*strong κ-c.c.*, for short) if $\mathbb{P}$ satisfies the κ-c.c. and for every κ-Suslin tree T, $\mathbb{P}$ does not add a cofinal branch of T.

Note 1.3.
(1) If there is no κ-Suslin tree, then the κ-c.c. is equivalent to the strong κ-c.c.
(2) For a poset $\mathbb{P}$, if $\mathbb{P} \times \mathbb{P}$ satisfies the κ-c.c., then $\mathbb{P}$ satisfies the strong κ-c.c.

Lemma 1.4. *If a poset $\mathbb{P}$ satisfies the μ-c.c. for some $\mu < \kappa$, then $\mathbb{P}$ satisfies the strong κ-c.c. In particular, every poset of size $< \kappa$ satisfies the strong κ-c.c.*

Proof. Suppose to the contrary that there is a κ-Suslin tree T such that $\Vdash_{\mathbb{P}}$"T has a cofinal branch $\dot{B}$". Let $T' = \{t \in T : p \Vdash_{\mathbb{P}}$"$t \in \dot{B}$"for some $p \in \mathbb{P}\}$. It is easy to check that T' is a downward closed subtree of T of height κ. Since $\mathbb{P}$ satisfies the μ-c.c. and $\mu < \kappa$, each level of T' has size $< \mu$. Now, by Kurepa's theorem, T' has a cofinal branch. Then this branch is a cofinal branch of T, this is a contradiction. □

The following is a main result of this note:

Lemma 1.5. *Let κ be a regular uncountable cardinal. Let $\mathbb{P}$ be an atomless poset which satisfies the strong κ-c.c. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a κ-closed poset (trivial poset is possible). Then $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-approximation property.*

Proof. Let $\tilde{\mathbb{Q}}$ be a term poset of $\dot{\mathbb{Q}}$, that is, $\tilde{\mathbb{Q}}$ is the set of all $\mathbb{P}$-names $\dot{q}$ with $\Vdash_{\mathbb{P}}$"$\dot{q} \in \dot{\mathbb{Q}}$". For $\dot{q}_0, \dot{q}_1 \in \tilde{\mathbb{Q}}$, define $\dot{q}_0 \leq \dot{q}_1$ if $\Vdash_{\mathbb{P}}$"$\dot{q}_0 \leq \dot{q}_1$ in $\dot{\mathbb{Q}}$". Since $\dot{\mathbb{Q}}$ is a name for a κ-closed poset, $\tilde{\mathbb{Q}}$ is κ-closed.

Let $\dot{x}$ be a $\mathbb{P}*\dot{\mathbb{Q}}$-name such that $\Vdash$"$\dot{x} \in V$". We say that a condition $\langle p, \dot{q}\rangle \in \mathbb{P}*\dot{\mathbb{Q}}$ *decides* $\dot{x}$ if there is y with $\langle p, \dot{q}\rangle \Vdash$"$\dot{x} = y$".

Claim 1.6. *Let τ be an ordinal and $\dot{f}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name such that $\Vdash$"$\dot{f} : \tau \to 2$ and $\dot{f}|x \in V$ for every $x \in ([\tau]^{<\kappa})^V$". Let $\langle p, \dot{q}\rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ and $x \in [\tau]^{<\kappa}$. Then there are $\dot{q}^* \leq \dot{q}$ and $F \subseteq {}^{x}2$ such that:*

(1) $|F| < \kappa$.
(2) For every $g \in F$, there is $p' \leq p$ such that $\langle p', \dot{q}^*\rangle \Vdash$"$\dot{f}|x = g$".
(3) For every $p' \leq p$, there are $p'' \leq p'$ and $g \in F$ such that $\langle p'', \dot{q}^*\rangle \Vdash$"$\dot{f}|x = g$".

Proof. It is easy to check that the set $\{p' \leq p : \exists \dot{q}' (\langle p', \dot{q}'\rangle \leq \langle p, \dot{q}\rangle$ and $\langle p', \dot{q}'\rangle$ decides $\dot{f}|x)\}$ is predense below p. Take a maximal antichain A which is contained in this set. Since $\mathbb{P}$ satisfies the κ-c.c., we know that $|A| < \kappa$. Then for each $r \in A$, there are $\dot{q}_r$ and g_r such that $\langle r, \dot{q}_r\rangle \leq \langle p, \dot{q}\rangle$ and $\langle r, \dot{q}_r\rangle \Vdash$"$\dot{f}|x = g_r$". Let $F = \{g_r : r \in A\}$ and one can take $\dot{q}^*$ such that $\dot{q}^* \leq \dot{q}$ and $r \Vdash$"$\dot{q}^* = \dot{q}_r$" for every $r \in A$. Then $\dot{q}^*$ and F work. $\square$[Claim]

In order to show that $\mathbb{P}*\dot{\mathbb{Q}}$ has the κ-approximation property, take $\langle p, \dot{q}\rangle \in \mathbb{P}*\dot{\mathbb{Q}}$, an ordinal τ, and a name $\dot{f}$ such that $\langle p, \dot{q}\rangle \Vdash$" $\dot{f} : \tau \to 2$ and $\dot{f}|x \in V$ for every $x \in ([\tau]^{<\kappa})^V$". Suppose to the contrary that $\langle p, \dot{q}\rangle \Vdash$"$\dot{f} \notin V$".

By induction on $\alpha < \kappa$, we would find $x_\alpha, \dot{q}_\alpha, F_\alpha$ $(\alpha < \kappa)$ such that:

(1) $x_\alpha \in [\tau]^{<\kappa}$ and $\langle x_\alpha : \alpha < \kappa\rangle$ is $\subseteq$-increasing.
(2) $\langle \dot{q}_\alpha : \alpha < \kappa\rangle$ is decreasing in $\tilde{\mathbb{Q}}$ and $\dot{q}_0 \leq \dot{q}$.
(3) $F_\alpha \subseteq {}^{x_\alpha}2$ and $|F_\alpha| < \kappa$.
(4) For every $g \in F_\alpha$, there is $p' \leq p$ such that $\langle p', \dot{q}_\alpha\rangle \Vdash$"$\dot{f}|x_\alpha = g$".
(5) For every $p' \leq p$ there are $p'' \leq p'$ and $g \in F_\alpha$ such that $\langle p'', \dot{q}_\alpha\rangle \Vdash$"$\dot{f}|x_\alpha = g$", i.e., the set $\{p' \leq p : \langle p', \dot{q}_\alpha\rangle \Vdash$"$\dot{f}|x_\alpha = g$" for some $g \in F_\alpha\}$ is predense below p.
(6) For every $g \in F_\alpha$, there are $g_0, g_1 \in F_{\alpha+1}$ such that $g \subseteq g_0, g_1$ and $g_0 \neq g_1$.

When $\alpha = 0$, pick an arbitrary $x_0 \in [\tau]^{<\kappa}$. Then we can find required $\dot{q}_0 \leq \dot{q}$ and F_0 by Claim 1.6.

Let $\alpha > 0$ and suppose $x_\beta, \dot{q}_\beta, F_\beta$ are defined for all $\beta < \alpha$.

Case 1: α is limit. We can find $x_\alpha \in [\tau]^{<\kappa}$ such that $x_\beta \subseteq x_\alpha$ for $\beta < \alpha$. Since $\tilde{\mathbb{Q}}$ is κ-closed, we can find $\dot{q}^* \leq \dot{q}_\beta$ for every $\beta < \alpha$. Then take $\dot{q}_\alpha \leq \dot{q}^*$ and F_α by Claim 1.6.

Case 2: α is successor, say $\alpha = \beta + 1$. Pick a maximal antichain $A \subseteq \mathbb{P}$ below p such that for every $p' \in A$ there is $g \in F_\beta$ such that $\langle p', \dot{q}_\beta\rangle \Vdash$"$\dot{f}|x_\beta = g$". Note that $|A| < \kappa$, and, for every $g \in F_\beta$, there is $p' \in A$ with $\langle p', \dot{q}_\beta\rangle \Vdash$"$\dot{f}|x_\beta = g$". Since $|A| < \kappa$ and $\langle p, \dot{q}_\beta\rangle \Vdash$"$\dot{f} \notin V$", we can find $x_\alpha \in [\tau]^{<\kappa}$ such that $x_\beta \subseteq x_\alpha$ for $\beta < \alpha$, but $\langle p', \dot{q}_\beta\rangle$ does not decide $\dot{f}|x_\alpha$ for every $p' \in A$.

Claim 1.7. *For each $p' \in A$, there are $p'_0, p'_1 \leq p'$, $g'_0, g'_1 : x_\alpha \to 2$, and $\dot{r} \leq \dot{q}_\beta$ such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{r}\rangle \Vdash$ "$\dot{f}|x_\alpha = g_i$".*

Proof. Since $\langle p', \dot{q}_\beta\rangle$ does not decide $\dot{f}|x_\alpha$, we can take $\langle p'_0, \dot{q}_0\rangle, \langle p'_1, \dot{q}_1\rangle \leq \langle p', \dot{q}_\beta\rangle$, and $g'_0, g'_1 : x_\alpha \to 2$ such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{q}_i\rangle \Vdash$"$\dot{f}|x_\alpha = g'_i$". We may assume that p'_0 is incompatible with p'_1; if p'_0 and p'_1 have a common extension p_2, take $p''_0, p''_1 \leq p_2$ such that $p''_0 \perp p''_1$ and replace p'_i by p''_i.

Now take $\dot{r} \leq \dot{q}_\beta$ such that $p''_i \Vdash$"$\dot{r} = \dot{q}_i$". Clearly p'_i, g'_i and $\dot{r}$ work. □[Claim]

For each $p' \in A$, pick $\dot{r}_{p'} \leq \dot{q}_\beta$ such that there are $p'_0, p'_1 \leq p'$, $g'_0, g'_1 : x_\alpha \to 2$ with $g'_0 \neq g'_1$ and $\langle p'_i, \dot{r}_{p'}\rangle \Vdash$"$\dot{f}|x_\alpha = g'_i$".

Then pick $q^* \leq q_\beta$ such that $p' \Vdash$"$\dot{q}^* = \dot{r}_{p'}$" for every $p' \in A$. Finally, take $\dot{q}_\alpha \leq \dot{q}^*$ and $F_\alpha \subseteq {}^{x_\alpha}2$ as in Claim 1.6. The following claim shows that x_α, $\dot{q}_\alpha$, and F_α work well:

Claim 1.8. *For each $g \in F_\beta$, there are $g_0, g_1 \in F_\alpha$ such that $g_0 \neq g_1$ and $g \subseteq g_0, g_1$.*

Proof. Take $p' \in A$ so that $\langle p', \dot{q}_\beta\rangle \Vdash$"$\dot{f}|x_\beta = g$". Then we can take $p'_0, p'_1 \leq p'$ and $g'_0, g'_1 : x_\alpha \to 2$ such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{q}^*\rangle \Vdash$"$\dot{f}|x_\alpha = g'_i$". Clearly $g \subseteq g'_0, g'_1$. By the choice of F_α and $\dot{q}_\alpha$, for each $i < 2$, one can take $p_i \leq p'_i$ and $g_i \in F_\alpha$ such that $\langle p_i, \dot{q}_\alpha\rangle \Vdash$"$\dot{f}|x_\alpha = g_i$". Since $\dot{q}_\alpha \leq \dot{q}^*$, each $\langle p_i, \dot{q}_\alpha\rangle$ is compatible with $\langle p'_i, \dot{q}^*\rangle$. This means that $g'_i = g_i$, so $g_0 \neq g_1$ and $g \subseteq g_0, g_1$. □[Claim]

Suppose $\dot{q}_\alpha, x_\alpha, F_\alpha$ are defined for $\alpha < \kappa$. Note that, for every $\alpha < \beta < \kappa$ and $g \in F_\beta$, we have $g|x_\alpha \in F_\alpha$; take $p' \leq p$ such that $\langle p', \dot{q}_\beta\rangle \Vdash$"$\dot{f}|x_\beta = g$". Then one can pick $p'' \leq p'$ and $h \in F_\alpha$ such that $\langle p'', \dot{q}_\alpha\rangle \Vdash$"$\dot{f}|x_\alpha = h$". $\langle p', \dot{q}_\beta\rangle$ is compatible with $\langle p'', \dot{q}_\alpha\rangle$. So $h = g|x_\alpha$.

Let $T = \bigcup_{\alpha<\kappa} F_\alpha$. T with the inclusion forms a κ-tree, and each node of T has at least two immediate successors.

Claim 1.9. *T has no antichain of size κ.*

Proof. For each $g \in T$, there are p_g and $\alpha_g < \kappa$ such that $\langle p_g, \dot{q}_{\alpha_g}\rangle \Vdash$"$\dot{f}|x_{\alpha_g} = g$". For g, g' in T, if g and g' are incompatible in T, then p_g is incompatible with $p_{g'}$ in $\mathbb{P}$. This means that if T has an antichain of size κ, then $\mathbb{P}$ also has an antichain of size κ. This is impossible, hence T does not have an antichain of size κ. □[Claim]

Hence T is a κ-Suslin tree. We finish the proof by showing the following claim, which contradicts the strong κ-c.c. of $\mathbb{P}$:

Claim 1.10. $p \Vdash_{\mathbb{P}}$ *"T has a cofinal branch".*

Proof. Take a $(V, \mathbb{P})$-generic G with $p \in G$ and work in $V[G]$. Let $\alpha < \kappa$. Since $\{p' \leq p : \langle p', \dot{q}_\alpha \rangle \Vdash \text{“}\dot{f}|x_\alpha = g\text{” for some } g \in F_\alpha\}$ is predense below p, we can find $p_\alpha \in G$ and $g_\alpha \in F_\alpha \subseteq T$ such that $\langle p_\alpha, \dot{q}_\alpha \rangle \Vdash \text{“}\dot{f}|x_\alpha = g_\alpha\text{”}$. Now, for $\alpha < \beta < \kappa$, p_α is compatible with p_β and $\dot{q}_\beta \leq \dot{q}_\alpha$. So $\langle p_\alpha, \dot{q}_\alpha \rangle$ is compatible with $\langle p_\beta, \dot{q}_\beta \rangle$. This means that $g_\alpha \subseteq g_\beta$, so $\{g_\alpha : \alpha < \kappa\}$ is a cofinal branch of T. □[Claim]

□

Note 1.11. If $\mathbb{P}$ satisfies the κ-c.c. but does not the strong κ-c.c., then $\mathbb{P}$ cannot have the κ-approximation property.

2. Applications

We consider some applications of Lemma 1.5.

Definition 2.1. Let κ be a regular uncountable cardinal and $\lambda \geq \kappa$ a cardinal. A set $X \subseteq \mathcal{P}_\kappa\lambda$ has the *strong tree property* if for every $\langle d_x : x \in X \rangle$ with $d_x \subseteq x$, if $|\{d_x \cap a : x \in X\}| < \kappa$ for every $a \in \mathcal{P}_\kappa\lambda$, then there is $D \subseteq \lambda$ such that for every $a \in \mathcal{P}_\kappa\lambda$ the set $\{x \in X : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa\lambda$.

Fact 2.2 (Viale-Weiss [3]). *(1) The following are equivalent:*
- *(a) $\mathcal{P}_\kappa\lambda$ has the strong tree property.*
- *(b) There is some unbounded set $X \subseteq \mathcal{P}_\kappa\lambda$ such that X has the strong tree property.*
- *(c) Every unbounded subset of $\mathcal{P}_\kappa\lambda$ has the strong tree property.*

(2) κ has the tree property if and only if $\mathcal{P}_\kappa\kappa$ has the strong tree property.
(3) κ is strongly compact if and only if κ is inaccessible and $\mathcal{P}_\kappa\lambda$ has the strong tree property for every $\lambda \geq \kappa$.
(4) Suppose Proper Forcing Axiom. Then $\mathcal{P}_{\omega_2}\lambda$ has the strong tree property for every $\lambda \geq \omega_2$.

Viale-Weiss [3] showed that for an inaccessible κ, if a standard κ-stage iteration satisfying the κ-c.c. forces that "$\kappa = \omega_2$ and Proper forcing axiom", then κ must be strongly compact in the ground model. The following is a slight improvement of their result.

Proposition 2.3. *Let κ be a regular uncountable cardinal. Suppose that there is a poset $\mathbb{P}$ which has the strong κ-c.c. and forces that "$\mathcal{P}_\kappa\lambda$ has the strong tree property for every $\lambda \geq \kappa$". Then $\mathcal{P}_\kappa\lambda$ has the strong tree property for every $\lambda \geq \kappa$ in the ground model.*

Proof. We check that $\mathcal{P}_\kappa\lambda$ has the strong tree property for every $\lambda \geq \kappa$. Fix $\lambda \geq \kappa$ and take $\langle d_x : x \in \mathcal{P}_\kappa\lambda\rangle$ such that $d_x \subseteq x$ and $|\{d_x \cap a : x \in \mathcal{P}_\kappa\lambda\}| < \kappa$ for every $a \in \mathcal{P}_\kappa\lambda$. Take a $(V, \mathbb{P})$-generic G and work in $V[G]$. In $V[G]$, $\mathcal{P}_\kappa^V\lambda$ is unbounded in $\mathcal{P}_\kappa\lambda$ since $\mathbb{P}$ satisfies the κ-c.c. By the strong tree property of $\mathcal{P}_\kappa^V\lambda$ in $V[G]$, we can find $D \subseteq \lambda$ such that $\{x \in \mathcal{P}_\kappa^V\lambda : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa\lambda$ for every $a \in \mathcal{P}_\kappa\lambda$. We see $D \in V$, this completes the proof. For each $a \in \mathcal{P}_\kappa^V\lambda$, there is $x \in \mathcal{P}_\kappa^V\lambda$ with $D \cap a = d_x \cap a \in V$. Thus, by the κ-approximation property of $\mathbb{P}$, we have $D \in V$. □

Next we look at the indestructibility of weak compactness.

Definition 2.4. Let κ be weakly compact. If every κ-directed closed forcing preserves the weak compactness of κ, then κ is said to be *indestructibly weakly compact.*

The existence of an indestructibly weakly compact cardinal is consistent (Laver [2]). The following theorem suggests that the consistency of the existence of an indestructibly weakly compact cardinal might be at least strongly compact cardinal.

Proposition 2.5. *Let κ be a regular uncountable cardinal. If there is a poset which satisfies the strong κ-c.c. and forces that "κ is indestructibly weakly compact", then κ is strongly compact.*

Proof. Take $\lambda \geq \kappa$. We see that $\mathcal{P}_\kappa\lambda$ has the strong tree property. Take $\langle d_x : x \in \mathcal{P}_\kappa\lambda\rangle$ with $d_x \subseteq x$ and $|\{d_x \cap a : x \in \mathcal{P}_\kappa\lambda\}| < \kappa$ for every $a \in \mathcal{P}_\kappa\lambda$.

Take a $(V, \mathbb{P})$-generic G, and a $(V[G], \mathrm{Col}(\kappa, \lambda))$-generic H. We work in $V[G][H]$. Fix a bijection $\pi : \lambda \to \kappa$. We know that $\{\pi``x : x \in \mathcal{P}_\kappa^V\lambda\}$ is unbounded in $\mathcal{P}_\kappa\kappa$. Since $\kappa$ is weakly compact in $V[G][H]$, by the tree property of $\kappa$, there is $C \subseteq \kappa$ such that $\{\pi``x \in \mathcal{P}_\kappa\kappa : \pi``(d_x) \cap a = C \cap a\}$ is unbounded for all $a \in \mathcal{P}_\kappa\kappa$. Put $D = \pi^{-1}``C$. Then for every $a \in \mathcal{P}_\kappa\lambda$, the set $\{x \in \mathcal{P}_\kappa^V\lambda : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa\lambda$. We know $D \in V$ since $\mathbb{P} * \mathrm{Col}(\kappa, \lambda)$ has the κ-approximation property by Lemma 1.5. □

References

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